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# VALUE OF INFORMATION FOR DECISION MAKERS WITH SUMEX AND LINEAR TIMES EXPONENTIAL UTILITY

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### Abstract

This paper analyzes the information acquisition problem in a two-action lottery setting. Information is evaluated using the buying price approach. We investigate the relationship between risk aversion and the value of information in the case of two one-switch utility function families: sumex, and linear plus exponential utility. We derive conditions under which there exists a monotonic relationship between the decision maker's risk tolerance and the value of information.

**Keywords:** Decision Analysis, Value of Information, One-switch Utility Functions, Sumex Utility, Linear times Exponential Utility

### Sumex ve Doğrusal Çarpı Üssel Fayda Fonksiyonlarına Göre Hareket Eden Karar Vericiler için Bilginin Değeri

### Özet

Bu makale iki seçenekli lotarya kararı verilen bir ortamda bilgi edinimi problemini ele almaktadır. Bilginin değeri alış fiyatı yaklaşımı ile hesaplanmıştır. Yaptığımız çalışmada riske duyarlılık ile bilginin değeri arasındaki ilişkiyi iki farklı tek değişimli fayda fonksiyonu ailesini kullanarak inceledik. Üzerinde çalışılan tek değişimli fayda fonksiyonu aileleri de sumex ve doğrusal artı üssel fayda fonksiyonu aileleridir. Bu bağlamda karar vericinin riske duyarlılığı ve bilginin değeri arasında hangi koşullar altında monotonik bir ilişki olduğu ortaya çıkarılmıştır.

**Anahtar Kelimeler:** Karar Analizi, Bilginin Değeri, Tek Değişimli Fayda Fonksiyonları, Sumex Fayda Fonksiyonu, Doğrusal çarpı Üssel Fayda Fonksiyonu

## 1. INTRODUCTION

The decision analysis approach to evaluating information is the use of utility functions. Decision makers are assumed to be expected utility maximizers; as such a decision under uncertainty involving many alternatives results in a choice that maximizes the expected value of the utility function, $u(x)$. It is quite common in economics and decision analysis domain to assume that decision makers are risk averse and that their utility functions are concave. The most widely accepted form of risk aversion measure is the risk aversion function, $r_u(x) = -u''(x)/u'(x)$ where $x$ is the monetary equivalent of all asset levels of the decision maker.

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Decision analysts have been studying the value of information problem extensively since Schlaifer (1959), Raiffa and Schlaifer (1961), Howard (1966) and Howard (1967). Various approaches have been proposed to evaluate information in the expected utility framework: expected utility increase, selling price, probability price, certainty equivalent and buying price approaches (see La Valle 1968 for an extensive comparative analysis of these approaches). We use the buying price approach in this paper that assigns a dollar value to information by measuring the decision maker's willingness to pay. One old and now resolved research question in this context was whether the intuitive argument that more risk averse decision makers are willing to bear a higher cost to acquire information. The answer is that a generic result does not exist (see Hilton 1981). Hence, the line of research on this topic focused on various well-known decision settings and sought to demonstrate a domain or problem specific monotonicity relation. Examples include Ohlson (1975) and Willinger (1989) that show a monotonic relationship if the probability distribution is small risk in the sense of Samuelson (1970).

The relationship between the value of information and risk aversion in a simple two-action setting is the theme of this paper. Information is evaluated using the buying price approach. A detailed description of the two-action decision environment is as follows: the decision maker chooses between a sure outcome and a risky prospect (or a lottery). If the decision maker accepts the lottery, then the terminal wealth is the initial wealth plus the outcome of the lottery. Conversely, a reject decision results in a deterministically known outcome. There exists much interest in two-action problems simply because of its practical validity extending to many real life problems such as replacement, investment choice and many others. In the context of information value, examples include Mehrez (1985), and Eeckhoudt and Godfroid (2000). The initial decision on the lottery can be either accept or reject, and at a point where the decision maker is indifferent between the two alternatives, the value of information is maximized (see Fatti et al. 1987 for the case of a risk neutral decision maker and Bickel 2008 for a risk averse decision maker).

This paper is an extension to the recent study by Abbas et al. (2013) where authors explore the behavior of buying price of information as a function of the risk attitude of the decision maker in a two-action decision setting. This paper shows that if the initial decision made by the decision maker is to reject the lottery without information acquisition, less risk averse decision makers are willing to pay more for information. In the case where the initial decision is to accept, a monotonic relation holds in the restricted sense. One-switch utility functions, which were largely characterized in a study by Bell (1988), are a focal discussion in the accept case because these utility functions possess some interesting properties that arguably best replicate the decision maker behavior in lotteries with monetary outcomes. A utility function is said to be one-switch, if the decision among two risky alternatives may change only once as the wealth level of the decision maker changes. A one-switch utility may belong toone of the four utility function families. Abbas et al. (2013) shows in the accept case that quadratic utility function family is the only family of one-switch utility functions in which value of information is monotonic with respect to the degree of risk aversion. In this paper, we therefore analyze special cases under which a monotonicity relation is observed for two of the one-switch utility function families: sumex and linear times exponential utility functions.

The paper is organized as follows. Section 2 provides the notation and definitions. In Section 3, we present our main results. Section 4 presents our concluding remarks.

## 2. MODEL FORMULATION

We consider a decision maker with a one-switch utility function $u(x)$ making decisions on a lottery $\Pi$ with either positive or negative outcomes. The decision maker may either accept or reject the lottery before observing the actual lottery outcome. We let $w$ be the decision maker's initial wealth. If the decision maker accepts the lottery, the terminal utility is $u(w+\Pi)$; if he rejects the lottery, the terminal utility is $u(w)$. The decision maker has the opportunity to acquire information on the occurrence of a number of mutually exclusive events $\{A_1, \dots, A_k\}$ where $\cup_{i=1}^{k} A_i$ includes all the lottery outcomes.

The buying price $B(w, I, u)$ of information $I$ generated by events $\{A_1, \dots, A_k\}$ for a decision maker with utility function $u$ and initial wealth $w$ is the maximum amount that the decision maker is willing to pay to acquire $I$. To simplify the notation, we use a shorthand form $B_u = B(w, I, u)$ throughout the paper unless an explicit notation is needed. $B_u$ satisfies

$$\max\{u(w), E[u(w+\Pi)]\} = \sum_i P(A_i) \cdot \max\{E[u(w+\Pi-B_u)|A_i], u(w-B_u)\}. \quad (1)$$

Following Bakır (2015), we define $d_u: \mathbb{R} \times \mathrm{B} \to R$ as the optimal decision function such that,

$$d_u(w, A) = \begin{cases} +1, & E[u(w+\Pi)|A] \geq u(w) \\ -1, & o.w. \end{cases}$$

where B denotes the collection of all possible events that the decision maker may acquire information on. In words, $d_u(w, A) = +1$ for any $A \in \mathrm{B}$ if the decision maker with initial wealth level $w$ accepts the lottery given that he knows the actual outcome lies in the set $A$. Using this decision function, we cluster the outcomes of the lottery in two sets: $\Gamma_u$ and its complement $\Gamma_u^c$. $\Gamma_u$ is defined as follows: $\Gamma_u(w, I) = \{\pi \in \mathbb{R}: d_u(w, A) = +1$ for $A \in \{A_1, \dots, A_k\}$ and $\pi \in A\}$. $\Gamma_u(w, I)$ is the union of mutually exclusive events that generate $I$ on which the decision maker accepts the lottery. Naturally, the complement event $\Gamma_u^c$ includes all the remaining outcomes of the lottery $\Pi$ that are not in $\Gamma_u$.



## 3. THE RESULTS ON SUMEX AND LINEAR TIMES EXPONENTIAL UTILITY FUNCTIONS

We first formally define one-switch utility functions referring to the main result in Bell (1988):

**Proposition 1** *(Bell 1988, Proposition 2) A utility function satisfies the one-switch rule if and only if it belongs to one of the following families:*

*(i) the quadratics,* $u(x) = ax^2 + bx + c$

*(ii) the sumex functions,* $u(x) = ae^{bx} + ce^{dx}$

*(iii) linear plus exponential,* $u(x) = ax + be^{cx}$

*(iv) linear times exponential,* $u(x) = (ax+b)e^{cx}$

As indicated earlier, this paper is concerned with the value of information behavior as a function of the degree of risk aversion of the decision maker with sumex and linear times exponential utility functions defined in Proposition 1. This behavior depends on the initial decision made, and a result that is relevant for all risk averse utility functions in the reject case is proved in Abbas et al. (2013). Therefore, we shift our focus on information acquired on lotteries that is initially accepted by the decision maker.

Past research on this question confirms that the aforementioned relationship is complex when the initial decision is to accept the lottery. Intuitively, the more risk averse decision maker is more likely to change his decision after acquiring a piece of information, so one may argue that more risk averse decision makers are willing to pay more for information acquisition. However, this does not always hold under the axioms of expected utility theory. Accordingly, we prove partial monotonicity results thathold for utility functions with some practical relevance for real life financial decisions.

### 3.1 Sumex Utility Function

The sumex utility function is another utility function which may be either decreasingly or increasingly risk averse depending on the signs of its parameters. Its most general form is $u(x) = ae^{bx} + ce^{dx}$ where $a$ and $c$ cannot be positive simultaneously because $u'' < 0$. It is possible to obtain a monotonicity result for the initial wealth level. However, the direction of monotonicity depends on the behavior of two exponential utility functions with risk coefficients $b$ and $d$ respectively. In this case, equation (1) can be rewritten for the sumex utility function to obtain,



$$ae^{bw}E[e^{b\Pi}] + ce^{dw}E[e^{d\Pi}] = P(\Gamma_u(w - B_u, I)) \cdot E[ae^{bw}e^{b(\Pi - B_u)} + ce^{dw}e^{d(\Pi - B_u)}|\Gamma_u(w - B_u, I)] + P(\Gamma_u^c(w - B_u, I)) \cdot (ae^{b(w - B_u)} + ce^{d(w - B_u)}).$$

(1)

Rearranging the above equation and using a shorthand notation $\Gamma_u = \Gamma_u(w - B_u, I)$ and $\Gamma_u^c = \Gamma_u^c(w - B_u, I)$, we obtain,

$$ae^{bw}\{E[e^{b\Pi}] - (P(\Gamma_u)E[e^{b(\Pi - B_u)}|\Gamma_u] + P(\Gamma_u^c)e^{-bB_u} + ce^{dw}\{E[e^{d\Pi}] - (P(\Gamma_u)E[e^{d(\Pi - B_u)}|\Gamma_u] + P(\Gamma_u^c)e^{-dB_u})\} = 0. \quad (2)$$

Equation (2) reveals that the buying price is determined based on how lotteries $\Pi$ and $Y = \Pi \cdot 1_{\Gamma_u} - B_u$ are compared by two exponential utility functions with coefficients $b$ and $d$. Depending on the signs, these exponential utility functions may either be risk averse or risk seeking. Note that, lottery $Y$ offers the outcome $\Pi$ minus $B_u$ on $\Gamma_u$ and $-B_u$ on $\Gamma_u^c$. In the light of these observations, westate the main proposition of this section. We use a function $sign(x)$ where $sign(x) = +1$ if $x > 0$ and $sign(x) = -1$ if $x < 0$.

**Proposition 2** *Consider a decision maker with a sumex utility function* $u(x) = au_1(x) + cu_2(x)$ *where* $u_1(x) = sign(b) \cdot e^{bx}$ *and* $u_2(x) = sign(d) \cdot e^{dx}$ *are two exponential utility functions. Assume that the initial decision on lottery* $\Pi$ *is 'accept' (i.e.,* $d_u(w, \mathbb{R}) = +1$*). Then, the buying price of information* $I, B(w, I, u)$ *exhibits the following behavior when* $r_u$ *is perturbed as a function of the initial wealth level* $w$*,*

*(i) Suppose* $u(x)$ *is decreasingly risk averse. If the more risk averse exponential utility function prefers* $Y$ *over* $\Pi$*, then the buying price is increasing in* $r_u$*. In the opposite case, the buying price is decreasing in* $r_u$*.*

*(ii) Suppose* $u(x)$ *is increasingly risk averse, and at least one of* $u_1$ *or* $u_2$ *is risk seeking. Then if the risk seeking exponential utility function prefers* $\Pi$ *over* $Y$*, then the buying price is increasing in* $r_u$*. In the opposite case, the buying price is decreasing in* $r_u$*.*

*(iii) Suppose* $u(x)$ *is increasingly risk averse and both* $u_1$ *and* $u_2$ *are risk averse. If* $|b| > |d|$ *and* $\Pi$ *is preferred over* $Y$*, then the buying price is increasing in* $r_u$*. If the direction of preference changes, the buying price is decreasing in* $r_u$*. On the other hand, if* $|d| > |b|$*, and* $\Pi$ *is preferred over* $Y$*, then the buying price is decreasing in* $r_u$*. If the direction of preference changes, the buying price is increasing in* $r_u$*.*

*Proof.* See Appendix A3.

A similar result could be proved for parameters *a* and *c*. The main reason for the need to introduce further conditions to ensure monotonicity is the switch in comparison of lotteries $\Pi$ and *Y*. Intuitively, one should expect that *Y* is more preferable to $\Pi$ as a decision maker with an exponential utility becomes risk averse, because *Y* offers a sure outcome on the set $\Gamma_u^c$. While our numerical analysis reveals that this line of argument works in an overwhelming majority parametric cases, counterexamples exist. In fact, this is also the main reason why there is no monotonic relationship between the buying price and risk aversion in the case of a zero-switch exponential utility function. The following example illustrates some of the results presented in Proposition 2.

**Example 1** First we illustrate an example where switch in the direction of monotonicity occurs in the case *(i)* of Proposition 2. Consider the below lottery with 8 outcomes as in Table I:

**Table I.** Lottery 1

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Probability | 0.125 | 0.125 | 0.050 | 0.200 | 0.125 | 0.125 | 0.125 | 0.125 |
| Outcome, $ | 10 | 3 | -1 | 12 | -1.8 | 15 | 2 | 5 |

We consider information on a single outcome where we learn whether a selected outcome in the lottery occurs. We use the utility function $u(x) = -ae^{-bx} - ce^{-dx}$ with parameter values *a* = 1, *b* = 1.01, *c* = 0.10, and *d* = 1. At the initial wealth level of *w* = 0, the buying price of information on outcome -1 is increasing whereas the buying price of information on outcome -1.8 is decreasing.

We also illustrate that a monotonicity result does not follow for parameters *b* and *d*. Using the same utility function, we first set parameter values as *a* = 1, *b* = 0.001, *c* = 10, and *d* = 1. At the initial wealth level of *w* = 5, the risk aversion function is increasing in *c* while the buying price of information on outcome -1 is decreasing. On the other hand, under parametric values *a* = 1, *b* = 0.1, *c* = 10, and *d* = 0.2, both quantities decrease as *c* is increased.

Next, using the same utility function with parameters *a* = 1, *b* = 1, *c* = 0.0001, and *d* = 0.001, the buying price of information on outcome -1 is decreasing in $r_u$ when *b* is increased. For the opposite direction, consider the lottery below as in Table II.

**Table II.** Lottery 2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Probability | 0.125 | 0.125 | 0.050 | 0.200 | 0.125 | 0.125 | 0.125 | 0.125 |
| Outcome, $ | 10 | -8 | -1 | 12 | -2 | 5 | 20 | 3 |

If we set parameters to *a* = 1, *b* = 0.1, *c* = 10, and *d* = 0.2, the risk aversion function and the buying price of information on outcome -8 are increasing in *b*. In sum, a monotonic relationship does not hold for *b* and *d*.

### 3.2 Linear Times Exponential Utility Function

The second one-switch utility function family considered in this paper is linear times exponential utility family with the general form $u(x) = (ax + b)e^{cx}$ . The risk aversion function is $r_u = (-bc^2 - 2ac - ac^2x)/(a + bc + acx)$, which is increasing in $x$ regardless of the signs of parameters. Linear times exponential utility function is the sum of $u_1(x) = be^{cx}$ which is the well known exponential utility function and $u_2(x) = axe^{cx}$ which behaves as a utility function only for a quite limited combination of parameter values. Therefore, unlike other one-switch utility functions, the behavior of the buying price cannot be characterized by a comparison of the original lottery $\Pi$ with another lottery (i.e. $\Pi \cdot 1_{r_u} - B_u$ as in the case of a sumex utility function) using a legitimate utility function. As such, we limit the discussion in this section to presentation of an example.

**Example 2** Consider the following lottery in Table III:

**Table III.** Lottery 3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Probability | 0.125 | 0.125 | 0.050 | 0.200 | 0.125 | 0.125 | 0.125 | 0.125 |
| Outcome, $ | 10 | 3 | -1 | 12 | -1.8 | 15 | 20 | 5 |

Assume the following parametric values for a linear times exponential utility function: $w = 0, a = -0.0001$ , $b = -0.45$, and $c = -1$. In this case, the risk aversion function is increasing in all the parameters and the initial wealth level. However, the buying price of information on $-1$ and $-1.8$ move in opposite direction when each of $w$, $a$, $b$ and $c$ are perturbed. This illustrates the lack of a monotonic relationship.

## 4. CONCLUSIONS

Information is a valuable commodity as it reduces uncertainty and leads to better decisions. It is a risky commodity as well because the decision maker seeking information does not know a priori the result of the information acquisition activity. The behavior of this risky commodity as a function of risk aversion has been studied in numerous papers in literature. In short, researchers have shown the initial decision has siginificant influence in information evaluation and in two-action settings, the relation becomes more complex in the case when the uncertain prospect is accepted. In this paper, we use the buying price approach to investigate the relation between the value of information and risk aversion in the accept case for two one-switch utility functions: sumex and linear times exponential. For sumex utility functions, we show that monotonicity requires strict conditions. Perhaps surprisingly, despite the strict conditions, we observe in our numerical analysis that a more risk averse decision maker values information more in an overwhelming majority of cases with an initial accept decision.

No monotonicity results were obtained for linear times exponential utility functions. One caveat to our results is that the monotonicity results are obtained as parameters that determine the risk aversion function are perturbed. However, the results do not directly characterize the relationship between the buying price and the associated parameters because the direction of monotonicity depends on how the risk aversion function changes as a function of those parameters.

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**Proof of Proposition 2:** There are several cases to consider for the sumex utility function. In what follows, we switch the signs of negative parameters for convenience.

*(i)*: There are three cases under which the sumex utility function is decreasingly risk averse. First is when all parameters are negative. After switching the signs of all parameters, the utility function can be rewritten in the first case as $u(x) = -ae^{-bx} - ce^{-dx}, a, b, c, d > 0$. Equation (2) can be rewritten as,



$$ae^{-bw}\{E[-e^{-b\Pi}] - (P(\Gamma_u)E[-e^{-b(\Pi-B_u)}|\Gamma_u]] + P(\Gamma_u^c)(-e^{-b(-B_u)}))\}$$
$$+ce^{-dw}\{E[-e^{-d\Pi}] - (P(\Gamma_u)E[-e^{-d(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(-e^{-d(-B_u)}))\} = 0. \quad (3)$$

Without loss of generality, assume $b > d$. If we multiply both sides of (3) with $e^{dw}$,

$$ae^{-(b-d)w}\{E[-e^{-b\Pi}] - (P(\Gamma_u)E[-e^{-b(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(-e^{-b(-B_u)}))\}$$
$$+c\{E[-e^{-d\Pi}] - (P(\Gamma_u)E[-e^{-d(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(-e^{-d(-B_u)}))\} = 0. \quad (4)$$

Since all parameters are positive, lottery $\Pi$ should be preferred over $Y$ by either $u_1(x) = -e^{-bx}$ or $u_2(x) = -e^{-dx}$. If $u_1(x) = -e^{-bx}$ prefers $\Pi$, then $\partial LHS/\partial w < 0$. Note that $\partial LHS/\partial B_u > 0$. Then an increase in $w$ implies an increase in $B_u$. Therefore, when the utility function $u(x) = -e^{-\max(b,d)x}$ prefers $\Pi$ over $Y$, then the buying price is higher for less risk averse decision makers. Conversely, using similar arguments, we can show that if $u(x) = -e^{-\max(b,d)x}$ prefers $Y$ over $\Pi$, then the buying price is higher as the risk aversion function increases.

The other cases where the risk aversion function is decreasing in the wealth level are $a, b < 0$ and $c, d > 0$ or $a, b > 0$ and $c, d < 0$. They are essentially identical, so we will illustrate the proof for one. Using positive parameters only, we proceed with the utility function $u(x) = -ae^{-bx} + ce^{dx}$. Since $u'' < 0$, $ab^2e^{-bx} > cd^2e^{dx}$. The risk aversion function is $r_u = (ab^2e^{-bx} - cd^2e^{dx})/(abe^{-bx} + cde^{dx})$. The buying price equation is,

$$ae^{-bw}\{E[-e^{-b\Pi}] - (P(\Gamma_u)E[-e^{-b(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(-e^{-b(-B_u)}))\}$$
$$= ce^{dw}\{(P(\Gamma_u)E[e^{d(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)e^{-dB_u}) - E[e^{d\Pi}]\}. \quad (5)$$

Both sides of (5) should have the same sign. If the risk averse exponential utility function $u_1(x) = -e^{-bx}$ prefers $\Pi$ over $Y$, then both sides are positive. In this case, $\partial LHS/\partial w < 0$ and

$\partial RHS/\partial w > 0$. Furthermore, $\partial LHS/\partial B_u > 0$ and $\partial RHS/\partial B_u < 0$. We can immediately conclude that an increase in $w$ results in an increase in $B_u B_u$. Conversely, similar arguments show that if $Y$ is preferred over $\Pi$ by $u_1(x) = -e^{-bx}$, then an increase in $w$ results in a decrease in $B_u$. In short, if the risk averse utility function $u_1(x) = -e^{-bx}$ prefers $\Pi$ over $Y$, then the buying price is decreasing as a function of the risk aversion function. In the opposite case, the buying price is higher for a more risk averse decision maker.

*(ii)*: There are four possible cases where the risk aversion function is increasing in the wealth level and at least one of the exponential utility functions is risk seeking. The proof of the first two and the last two are identical. First two cases are $a < 0, b, c, d > 0$ and $c < 0, a, b, d > 0$. Using only positive parameters, we consider $u(x) = -ae^{bx} + ce^{dx}$. Usual $u' > 0$ and $u'' < 0$ conditions imply $cde^{dx} > abe^{bx}$ and $cd^2e^{dx} < ab^2e^{bx}$ for all terminal wealth levels $x$. Since all parameters are positive, $b > d$ should follow. The risk aversion function is $r_u = (ab^2e^{bx} - cd^2e^{dx})/(-abe^{bx} + cde^{dx})$. Equation (2) can be modified in this case to obtain,

$$ae^{bw}\{E[e^{b\Pi}] - (P(\Gamma_u)E[e^{b(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)e^{-bB_u})\}$$
$$= ce^{dw}\{E[e^{d\Pi}] - (P(\Gamma_u)E[e^{d(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)e^{-dB_u})\}. \quad (6)$$

Since $b > d, b \cdot LHS > d \cdot RHS$. This implies $\partial LHS/\partial w > \partial RHS/\partial w$ if both derivatives are positive (i.e., $u_1(x) = e^{bx}$ and $u_2(x) = e^{dx} u_2(x) = e^{dx}$ prefer $\Pi\Pi$ over $Y$). Otherwise, $\partial LHS/\partial w < \partial RHS/\partial w < 0$ $\partial LHS/\partial B_u = abe^{b(w-B_u)}[P(\Gamma_u)E[e^{b\Pi}|\Gamma_u] + P(\Gamma_u^c)]$ $\partial LHS/\partial B_u = abe^{b(w-B_u)}[P(\Gamma_u)E[e^{b\Pi}|\Gamma_u] + P(\Gamma_u^c)]$ , and . We already know that $cde^{dx} > abe^{bx}$ for all terminal wealth levels *x*. Therefore, $\partial RHS/\partial B_u > \partial LHS/\partial B_u > 0$. Now, if $\partial LHS/\partial w, \partial RHS/\partial w > 0$, then an increase in $w$ results in $LHS > RHS$. Therefore, $B_u$ should be increased as well because $RHS$ is increasing faster as $B_u$ is increased. Conversely, if $\partial LHS/\partial w, \partial RHS/\partial w < 0$, an increase in $w$ renders a decrease in $B_u$.

The last two cases are $a, c, b < 0, d > 0$ and $a, c, d < 0, b > 0$. Without loss of generality, we use $u(x) = -ae^{-bx} - ce^{dx}$, where $a, b, c, d > 0$ and $abe^{-bx} > cde^{dx}$ (because $u' > 0$). Also, $r_u = (ab^2e^{-bx} + cd^2e^{dx})/(abe^{-bx} - cde^{dx})$. The buying price, $B_u$ solves the equation,

$$ae^{-bw}\{E[-e^{-b\Pi}] - (P(\Gamma_u)E[-e^{-b(\Pi-B_u)}|\Gamma_u]] + P(\Gamma_u^c)(-e^{-b(-B_u)}))\}$$
$$= ce^{dw}\{E[e^{d\Pi}] - (P(\Gamma_u)E[e^{d(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)e^{-dB_u})\}. \quad (7)$$

The equation (7) suggests that if the risk seeking utility function $u_1(x) = e^{dx}$ prefers $\Pi$ over $Y$, then both sides are positive. Then, clearly $\partial LHS/\partial w < 0$ and $\partial RHS/\partial w > 0$. We also calculate $\partial LHS/\partial B_u = ab(P(\Gamma_u)E[e^{-b(w+\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(e^{-b(w-B_u)}))$ and $\partial RHS/\partial B_u = cd(P(\Gamma_u)E[e^{d(w+\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(e^{d(w-B_u)}))$. We know $abe^{-bx} > cde^{dx}$ for all terminal wealth levels *x*, which implies $0 < \partial RHS/\partial B_u < \partial LHS/\partial B_u$. Hence, an increase in $ww$ results in an increase in $B_u$. We could similarly argue that if $Y$ is preferred over $\Pi$ by $u_1(x) = e^{dx}$, then an increase in $w$ results in a decrease in $B_u$. Under (ii), when the risk seeking exponential utility function $u(x) = e^{dx}$ prefers $\Pi$ over $Y$, then the buying price is increasing as the risk aversion function increases as a function of $w$. In the opposite case, the buying price is lower for a more risk averse decision maker.

*(iii)*: There are two cases of parametric combinations: $a, d, b < 0, c > 0$ and $c, d, b < 0, a > 0$. Since their proofs are identical, we consider only the first case. With appropriate sign changes, the utility function becomes $u(x) = -ae^{-bx} + ce^{-dx}$ (i.e.,$a, b, c, d > 0$). Both $u' > 0$ and $u'' < 0$ impose restrictions, $abe^{-bx} > cde^{-dx}$ and $ab^2e^{-bx} > cd^2e^{-dx}$. The risk aversion function is $r_u = (ab^2e^{-bx} - cd^2e^{-dx})/(abe^{-bx} - cde^{-dx})$. $B_u$ satisfies,

$$ae^{-bw}\left\{E[-e^{-b\Pi}] - \left(P(\Gamma_u)E\left[-e^{-b(\Pi-B_u)}|\Gamma_u\right] + P(\Gamma_u^c)\left(-e^{-b(-B_u)}\right)\right)\right\}$$
$$= ce^{-dw}\{E[-e^{-d\Pi}] - (P(\Gamma_u)E[-e^{-d(\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(-e^{-d(-B_u)}))\}.$$

The comparison between $\Pi$ and $Y$ should yield an identical result for the risk averse exponential utility functions $u_1(x) = -e^{-bx}$ and $u_2(x) = -e^{-dx}$. If $\Pi$ is preferred over $Y$ by both, then $\partial LHS/\partial w < 0$ and $\partial RHS/\partial w < 0$. Furthermore, if $b > d$, $\partial LHS/\partial w < \partial RHS/\partial w$ which implies $LHS$ is decreasing faster as $w$ increases. As far as $B_u$ is concerned $\partial LHS/\partial B_u = ab(P(\Gamma_u)E[e^{-b(w+\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(e^{-b(w-B_u)}))$ $\partial LHS/\partial B_u = ab(P(\Gamma_u)E[e^{-b(w+\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(e^{-b(w-B_u)}))$ and $\partial RHS/\partial B_u = cd(P(\Gamma_u)E[e^{-d(w+\Pi-B_u)}|\Gamma_u] + P(\Gamma_u^c)(e^{-d(w-B_u)}))$. We know $abe^{-bx} > cde^{-dx}$ for all terminal wealth levels *x*, which implies $0 < \partial RHS/\partial B_u < \partial LHS/\partial B_u$. From that, we see an increase in $B_u$ resulting from an increase in $w$. If $b < d$, then $\partial LHS/\partial w > \partial RHS/\partial w$, and $B_u$ needs to be decreasing.

Going back, now assume $Y$ is preferred over $\Pi$ by both exponential utility functions. If $b > d$, then $\partial LHS/\partial w > \partial RHS/\partial w > 0$. Thus $B_u$ needs to be decreasing. If $b < d$, then an increase in $w$ results in an increase in $B_u$. As a result, we obtain the directional relationship stated in the proposition for this case.